Majorization Problem on a Unified Class of Analytic Functions Using Certain Fractional Differential Operator

Abstract

Main purpose of this paper is to investigate majorization problem involving starlike functions of complex order belonging to a new unified class $S_{p,q}^{\lambda,j,m}\left[A,B;\nu\right]$ of functions which are multivalent analytic in the open unit disk $U=\left\{z:z\in\mathbb{C}:\left|z\right|<1\right\}$ defined by means of multi order fractional derivatives. Here we will also point out some interesting consequences of our main result.

Keywords: Majorization, Starlike Function, Hadmard Product, Differential Operator

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Introduction

Let $A\left(p\right)$ denote the class of multivalent functions $f(z)$

(1.1) $$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k, \qquad a_k\geq 0 \qquad \left(p\in\mathrm{N}\right)$$

That are analytic in the open unit disk $U=\left\{z:z\in\mathbb{C}:\left|z\right|<1\right\}$.

A function $f\in A(p)$ is said to be *p*–valently starlike of order α $\left(0\leq\alpha<p\right)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad \left(z\in U,0\leq\alpha<p\right)$$

The classes of all such functions are denoted by $S_p^*\left(\alpha\right)$.

Aim of the Study

The aim of this paper is to investigate majorization problem involving starlike functions of complex order. Here we will also point out some interesting consequences of our main result.

Let $K_p\left(\alpha\right)$ denote the class of all those functions which are *p* -valently convex of order α $\left(0\leq\alpha<p\right)$ in *U*, if and only if

$$\mathrm{Re}\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>\alpha \qquad \left(z\in U,0\leq\alpha<p\right)$$

We note that $S_p^*\left(0\right)=S_p^*$, $K_p\left(0\right)=K_p$ are the well-known classes of *p*-valent starlike,*p*-valent convex in *U*. Also $S_1^*=S^*$, $K_1=K$ are the usual classes of univalent starlike and convex in *U*.

Let $g\left(z\right)$ be analytic in open unit disk $U=\left\{z:z\in\mathbb{C},\left|z\right|<1\right\}$. We say that f is majorized by g in U By MacGreogor [3] and write

(1·2) $$f\left(z\right)\ll g\left(z\right), \qquad \left(z\in U\right)$$

If there exists a function ϕ, analytic in U such that $\left|\phi(z)\right| \le 1$ as

(1·3) $$f(z) = \phi(z) g(z) \qquad (z \in U)$$

This is closely related to the concept of quasi-convolution between analytic functions. For functions $f_j \in A(p)$ given by

(1·4)

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \qquad \left(a_{k,j} \ge 0; j = 1,2; p \in \mathrm{N}\right)$$

Modified Hadamard product (quasi-convolution) of f_1 and f_2 defined by (1.5)

$$\left(f_1 * f_2\right)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = \left(f_2 * f_1\right)(z)$$

For two functions f and g analytic in U we say that f is subordinate to g in U and write $f(z) \prec g(z)$

If there exists a Schwarz function $\omega(z)$, which is analytic in U with $\omega(0) = 0$ and $\left| \omega(z) \right| < 1,$

such that

$$f(z) = g\left(\omega(z)\right), \qquad (z \in U)$$

Indeed, it is known that

$f(z) \prec g(z) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$

Let $f^{(q)}$ denote the q^{th} order ordinary differential of function $f \in A(p)$, that is,

(1·6)

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q},$$

Where,

$$(p > q;\ p \in \mathrm{N};\ q \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}), \qquad (z \in U)$$

.

Now we introduce the new differential operator $D f^{(q)}(z)$ as follows

(1·7)

$$Df^{(q)}(z) = \left\{1 - \lambda(p-q)\right\} f^{(q)}(z) + \lambda z \left(f^{(q)}(z)\right)', \quad \lambda \ge 0$$

and $D^m f^{(q)}(z) = D\left(D^{m-1}\right) f^{(q)}(z)$

(1·8)

$$D^m f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left\{1 - \lambda(p-k)\right\}^m a_k z^{k-q}$$

Putting $m = 0$ we get

$$D^0 f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left\{1 - \lambda(p-k)\right\}^0 a_k z^{k-q}$$

$$= f^{(q)}(z)$$

Putting $m = 0,\ q = 0$ $D^0 f^0(z) = f(z)$

And we get the identity

(1·9)

$$\lambda z \left[D^m f^{(q)}(z)\right]' = D^{m+1} f^{(q)}(z) - \left\{1 - \lambda(p-q)\right\} D^m f^{(q)}(z)$$

$$\left(-\infty < m < (p-q), (z \in U), (0 \le \lambda \le 1)\right)$$

Remarks

1. For $p = 1$ and $q = 0$, it reduces to well known Al-oboudi operator
2. For $p = \lambda = 1$ and $q = 0$, it reduces to Salagean [6] operator.

Definition

A function $f(z) \in A(p)$ is said to be in the class $S_{p,q}^{\lambda,j,m}(A,B;\nu)$ of p-valent functions of complex order $\nu \ne 0$ in U if and only

(1·10)

$$\mathrm{Re}\left\{1 + \frac{1}{\nu}\left(z \frac{\left(D^m f^{(q)}(z)\right)^{j+1}}{\left(D^m f^{(q)}(z)\right)^j} - (p-q-j)\right)\right\} > 0$$

$$\left(z \in U, -1 \le B < A \le 1, p \in \mathbb{N}, j, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \nu \in \mathbb{C} - \{0\}, -\infty < m < (p-q)\right)$$

The family $S_{p,q}^{\lambda,j,m}(A,B;\nu)$ unifies various well known classes of analytic univalent and multivalent functions. We list a few of them.

1. $S_{p,0}^{\lambda,0,0}(1,-1;\nu) = S_p(\nu),$ $\left(\nu \in \mathbb{C} - \{0\}\right)$, by Naser [4]
2. $S_{p,0}^{\lambda,1,0}(1,-1;\nu) = K_p(\nu),$ $\left(\nu \in \mathbb{C} - \{0\}\right)$, by Naser [4]
3. $S_{p,0}^{\lambda,j,0}(1,-1;\nu) = S_j(p,\nu),$ by Akbulut [1]

4. $S_{p,q}^{\lambda,0,m}(1,-1;\nu)=S_{p,q}^{\lambda,m}(\nu)$, by Goyal[2]

5. $S_{1,0}^{0,0,0}(1,-1;1-\alpha)=S^{*}(\alpha)$, for $0\le\alpha<1$, by Shrivastava[7].

Majorization Problem for the Class $S_{p,q}^{\lambda,j,m}(A,B;\nu)$

Theorem2.1

Let the function $f\in A(p)$ and supposes that $g\in S_{p,q}^{\lambda,j,m}(A,B;\nu)$. If $\left(D^m f^{(q)}(z)\right)^j$ is majorized by $\left(D^m g^{(q)}(z)\right)^j$ in *U*, then $\left(D^{m+1} f^{(q)}(z)\right)^j$ is majorized by $\left(D^{m+1} g^{(q)}(z)\right)^j$ i.e.

(2·1)

$$\left|\left(D^{m+1} f^{(q)}(z)\right)^j\right|\le\left|\left(D^{m+1} g^{(q)}(z)\right)^j\right|$$

For $\left(|z|\le r_0\right)$

Where $r_0=r_0(\lambda,\nu,A,B)$ is the root of the followingeq..

(2·2)

$$r^3\left(|\lambda\nu(A-B)+B|\right)-r^2\left(1+2\lambda|B|\right)-r\left(2\lambda+|\lambda\nu(A-B)+B|\right)+1=0$$

$$\left(-1\le B<A\le1,\nu\in\mathbb{C}-\{0\}\right)$$

Proof: Since $g\in S_{p,q}^{\lambda,j,m}(A,B;\nu)$

(2·3)

$$h(z)=1+\frac{1}{\nu}\left(z\frac{\left(D^m g^{(q)}(z)\right)^{j+1}}{\left(D^m g^{(q)}(z)\right)^j}-(p-q-j)\right)$$

$$\left(\nu\in\mathbb{C}-\{0\},\,p\in\mathbb{N},q,\,j\in\mathbb{N}_0\text{and } q>m,(p-q)>j\right)$$

Then, $\operatorname{Re}\{h(z)\}>0,\quad(z\in U)$ and

(2·4) $$\operatorname{Re}\{h(z)\}=\frac{1+A\omega(z)}{1+B\omega(z)},$$

Where ω is analytic in U with $\omega(0)=0$

And $|\omega(z)|\le1\qquad(z\in U)$

(2·5)

$$\left\{1+\frac{1}{\nu}\left(z\frac{\left(D^m g^{(q)}(z)\right)^{j+1}}{\left(D^m g^{(q)}(z)\right)^j}-(p-q-j)\right)\right\}=\frac{1+A\omega(z)}{1+B\omega(z)}$$

By simplification, we get

(2·6)

$$z\frac{\left(D^m g^{(q)}(z)\right)^{j+1}}{\left(D^m g^{(q)}(z)\right)^j}=\frac{\left(\nu(A-B)+B(p-q-j)\right)\omega(z)+(p-q-j)}{1+B\omega(z)}$$

Differentiating eq. (1.9)j[th] times, we get

(2·7)

$$\lambda z\left(D^m g^{(q)}(z)\right)^{j+1}=\left(D^{m+1} g^{(q)}(z)\right)^j-\{1-\lambda(p-q-j)\}\left(D^m g^{(q)}(z)\right)^j$$

By simple calculations, we get

(2.8)

$$\left(D^{m+1} g^{(q)}(z)\right)^j=\frac{[\lambda\nu(A-B)\omega(z)]+1}{1+B\omega(z)}\left(D^m g^{(q)}(z)\right)^j$$

Since $\left(D^m f^{(q)}(z)\right)^j$ is majorized by $\left(D^m g^{(q)}(z)\right)^j$ in the unit disk*U*, there exists a function $\psi(z)$, analytic in *U* such that $|\psi(z)|\le1$

(2·9) $$\left(D^m f^{(q)}(z)\right)^j=\psi(z)\left(D^m g^{(q)}(z)\right)^j$$

Differentiatingeq. (2.9) and multiplying both sidesby $'z'$, we get

(2·10)

$$z\left(D^m f^{(q)}(z)\right)^{j+1}=z\psi'(z)\left(D^m g^{(q)}(z)\right)^j+\psi(z)z\left(D^m g^{(q)}(z)\right)^{j+1}$$

Then by Nehari [5] inequality

(2·11) $$|\psi'(z)|\le\frac{1-|\psi(z)|^2}{1-|z|^2},$$

$(z\in U)$

U

Using eqs. (2·7), (2·8) and (2·11) in (2.10), we get (2·12)

$$\left|\left(D^{m+1} f^{(q)}(z)\right)^j\right|\le\left(\psi(z)+|z|\left(\frac{1-|\psi(z)|^2}{1-|z|^2}\right)\frac{\lambda\left(1+|B||z|\right)}{\left(1-|\lambda\nu(A-B)+B||z|\right)}\right)\left|\left(D^{m+1} g^{(q)}(z)\right)^j\right|$$

Which, upon setting $|z|=r$ and $|\psi(z)|=\rho,(0\le\rho\le1)$ leads us to the inequality (2·13)

$$\left|\left(D^{m+1}f^{(q)}(z)\right)^j\right|\le\left(\frac{\Phi(\rho)}{(1-r^2)\left(1-|\lambda\nu(A-B)+B||r|\right)}\right)\left|\left(D^{m+1}f^{(q)}(z)\right)^j\right|$$

where (2.14) $\Phi(\rho)=-\lambda r\left(1+|B|r\right)\rho^2+\left(1-r^2\right)\left[1-|\lambda\nu(A-B)+B|r\right]\rho+\lambda r\left(1+|B|r\right)$

takes its maximum value at $\rho=1$, with $r_0=r_0(\lambda,\nu,A,B)$ where r_0 is given by eq. (2.2).

Furthermore, if $0\le\rho\le r_0(\lambda,\nu,A,B)$, then the function $\varphi(\rho)$ defined by

(2.15) $\varphi(\rho)=-\lambda\sigma\left(1+|B|\sigma\right)\rho^2+\left(1-\sigma^2\right)\left[1-|\lambda\nu(A-B)+B|\sigma\right]\rho+\lambda\sigma\left(1+|B|\sigma\right)$

is seen to be an increasing function on the interval $0\le\rho\le1$, so that (2.16)

$$\varphi(\rho)\le\varphi(1)=\left(1-\sigma^2\right)\left[1-|\lambda\nu(A-B)+B|\sigma\right]\rho$$

$$\left(0\le\sigma\le r_0(\lambda,\nu,A,B),(0\le\rho\le1)\right)$$

Hence upon setting, $\rho=1$ in eq. (2.15), we conclude that eq. (2.1) of theorem (2.1) holds true for $|z|\le r_0(\lambda,\nu,A,B)$,

where r_0 is given by eq. (2.2).

This completes the Proof.

Corollary 2.1

Let the function $f\in A(p)$ and suppose that $g\in S_{p,q}^{\lambda,j,m}(1,-1;\nu)$. If $\left(D^m f^{(q)}(z)\right)^j$ is majorized by $\left(D^m g^{(q)}(z)\right)^j$ in *U* then

$$\left|\left(D^{m+1}f^{(q)}(z)\right)^j\right|\le\left|\left(D^{m+1}g^{(q)}(z)\right)^j\right|,$$

$$(|z|\le r_1)$$

Where value of $r_1=r_1(\lambda,\nu)$ is given by the eq.

$$r^3\left(|2\lambda\nu-1|\right)+r^2(1+2\lambda)-r\left(2\lambda+|2\lambda\nu-1|\right)+1=0$$

Corollary 2.2: Let the function $f\in A(p)$ and suppose that $g\in S_{p,q}^{1,j,m}(1,-1;\nu)$. If $\left(D^m f^{(q)}(z)\right)^j$ is majorized by $\left(D^m g^{(q)}(z)\right)^j$ in U then

$$\left|\left(D^{m+1}f^{(q)}(z)\right)^j\right|\le\left|\left(D^{m+1}g^{(q)}(z)\right)^j\right|,$$

$$(|z|\le r_2)$$

Where value of $r_2=r_2(\nu)$ given by the eq.

$$r^3\left(|2\nu-1|\right)+3r^2-r\left(2+|2\nu-1|\right)+1=0$$

Conclusion

This paper presented majorization property and derive some results from main result.

References

1. *Akbulut, S., Kadioglu, E., Ozdemir, M., On the Subclass of p-Valently Functions, Appl. Math. Comput.,147(1) (2004), 89-96*
2. *Goyal, S.P., Goswami, P., Majorization for certain Classes of Analytic Functions defined by Fractional Derivatives,Appl. Math. Lett.,22(12) (2009), 1855-1858.*
3. *Macgreogor, T.H., Majorization by Univalent Functions, Duke Math. J.,34 (1967), 95-102.*
4. *Naser, M.A., Aouf, M.K., Starlike Function of Complex Order, J. Natur. Sci. Math., 25(1985), 1-12.*
5. *Nehari, Z., Conformal Mapping, Macgraw-Hill Book Company, New York, Toronto andLondon,(1952).*
6. *Salagean, G.S., Subclasses of Univalent Functions, Lecture Notes InMath., Springer, Berlin, 1013 (1983), 362-372.*
7. *Srivastava, H.M., Owa, S., Chatterjee, S.K., A Note on certain Classes of Starlike Functions, Rend. Sem. Mat. Univ Padova, 77(1987), 115-124.*